Christoffel symbols are tensors

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Goals

Prove that the Christoffel symbols are vectors and, therefore, they can be thought of as rank-1 tensors (but not necessarily) [3].

1 Proof

Let us consider, in the pseudo-Riemannian manifold $(\mathcal{M}, \boldsymbol{g})$ with $\boldsymbol{g}$ metric tensor, the transformation

$$\partial_l \boldsymbol{e}_m = \alpha \partial_i \boldsymbol{e}_j + \beta \partial_\mu \boldsymbol{e}_\nu = \alpha \Gamma^k_{ij} \boldsymbol{e}_k + \beta \Gamma^k_{\mu\nu} \boldsymbol{e}_k = (\alpha \Gamma^k_{ij} + \beta \Gamma^k_{\mu\nu}) \boldsymbol{e}_k = \Gamma^k_{lm} \boldsymbol{e}_k$$

with $\boldsymbol{e}_m = \partial_m$ tangent space generator vector $T_{\boldsymbol{x}}\mathcal{M}(\mathbb{K})$ and $\alpha, \beta \in \mathbb{K}$. It is clear that, since Christoffel symbols depend on the derivatives of the metric tensor, if $\mathcal{M}$ is flat (i.e. $R^i_{jkl} = 0$) it is possible to find a coordinate system such that $\Gamma^k_{lm} = 0$ i.e. we have proven the **existence of the Identity element of addition**. Thus, if the various Γ^k_{lm} will prove to be vectors, this transformation will serve as a *subspace criterion*.

The following conditions also apply:

1. $(\partial_i \boldsymbol{e}_j + \partial_\mu \boldsymbol{e}_\nu) + \partial_l \boldsymbol{e}_m = \partial_i \boldsymbol{e}_j + (\partial_\mu \boldsymbol{e}_\nu + \partial_l \boldsymbol{e}_m)$;
2. $R^i_{jkl} = 0 \implies \exists \{x^i\} \mid \partial_i \boldsymbol{e}_j = 0 \, : \, \partial_\mu \boldsymbol{e}_\nu + 0 = 0 + \partial_\mu \boldsymbol{e}_\nu = \partial_\mu \boldsymbol{e}_\nu$;
3. $\partial_i \boldsymbol{e}_j + \partial_i(-\boldsymbol{e}_j) = \partial_i \boldsymbol{e}_j - \partial_i \boldsymbol{e}_j = 0$;
4. $\partial_i \boldsymbol{e}_j + \partial_\mu \boldsymbol{e}_\nu = \partial_\mu \boldsymbol{e}_\nu + \partial_i \boldsymbol{e}_j$;

Calling $\boldsymbol{\Gamma}$ the set of Γ^k_{lm} and remembering that $\Gamma^k_{lm} \boldsymbol{e}_k \cdot \boldsymbol{e}^h = \Gamma^k_{lm} \delta^h_k = \Gamma^h_{lm}$, the pair $(\boldsymbol{\Gamma}, +)$ is an abelian group. Furthermore, we have:

a. $\alpha(\partial_i \boldsymbol{e}_j + \partial_\mu \boldsymbol{e}_\nu) = \alpha \partial_i \boldsymbol{e}_j + \alpha \partial_\mu \boldsymbol{e}_\nu$

b. $(\alpha \hat{+} \beta) \partial_i \boldsymbol{e}_j = \alpha \partial_i \boldsymbol{e}_j + \beta \partial_i \boldsymbol{e}_j$

c. $1 \partial_i \boldsymbol{e}_j = \partial_i \boldsymbol{e}_j 1 = \partial_i \boldsymbol{e}_j$

d. $\alpha(\beta \partial_i \boldsymbol{e}_j) = (\alpha * \beta) \partial_i \boldsymbol{e}_j$

And so, we have shown that $\boldsymbol{\Gamma}$ acquires $\boldsymbol{\Gamma}(\mathbb{K})$ structure i.e. a vector space V on the field $\mathbb{K}$ $\square$ [2].

But yet, it is well known that Christoffel symbols do not constitute rank-3 tensors as they not satisfy the request

$$T'^{k}_{lm}(\boldsymbol{y}) = \frac{\partial x^i}{\partial y^l}\frac{\partial x^j}{\partial y^m}\frac{\partial y^k}{\partial x^h}T^h_{ij}(\boldsymbol{x}); [1]$$

the solution to this apparent paradox lies in the fact that Christoffel symbols are vectors in their vector space $\mathbf{\Gamma}(\mathbb{K})$ i.e. they can be formalized as rank-1 tensors in $\mathbb{K}^{64}(\mathbb{K})$ as well as they represent coefficients on the manfiold $\mathcal{M}$ $\square$.

2 Comments

Too often many authors tend to confuse the geometric notion of vector with the algebraic one; in fact, using the latter, there is no request that a vector respond to any "metric local property" since no geometry is involved. Generally speaking, a vector is simply an element that respects the **algebraic properties** listed above, nothing more; certainly it is necessary to specify where these quantities represent vectors (and where they don't).

References

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